# Changing the Order for a Triple Integral

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The purpose of this note is to compute the volume of a certain solid in $\mathbb{R}^3$ in six different ways, corresponding to the six possible orders of integration in a triple integral. Let's consider the solid contained in the first octant, and bounded by the plane $y = 1 - x$ and the surface $z = 1 - x^2$; a graph of this solid was done in class today, and it's also problem 34 in Section 15.7 of the textbook. This can be described by the inequalities

$$\begin{aligned} 0 &\le z \le 1 - x^2 \\ 0 &\le y \le 1 - x \\ 0 &\le x \le 1 \end{aligned}$$

Note that the "corners" of the object lie at the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. We'll now compute the volume via $V = \int\int\int_E dV$ with each possible order:

- The easiest way to order it is either $dzdydx$ or $dydzdx$, since we already have the inequalities describing the bounds. Let's do $dzdydx$ first: we can write

$$\begin{aligned} V &= \int_0^1 \int_0^{1-x} \int_0^{1-x^2} dz\,dy\,dx \\ &= \int_0^1 \int_0^{1-x} 1 - x^2\,dy\,dx \\ &= \int_0^1 (1 - x^2)(1 - x)dx \\ &= \int_0^1 1 - x - x^2 + x^3 dx \\ &= 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{10}{24} = \boxed{\frac{5}{12}} \end{aligned}$$

- It's just as easy to do $dydzdx$, giving

$$V = \int_0^1 \int_0^{1-x^2} \int_0^{1-x} dz\,dy\,dx$$

  Computing this also gives $5/12$, in essentially the same way.

- Let's next do order $dydxdz$. The $z$ bounds are easiest, since we have $0 \le z \le 1$. Now for each fixed $z$, we need to determine what appropriate bounds on $x$ and $y$ are (where the bounds on $x$ can *only* depend on $z$). Solving for $x$ in terms of $z$, we now have the inequality

$$0 \le x \le \sqrt{1 - z}$$

  Furthermore, we have $0 \le y \le 1 - x$, so our integral ought to be

$$V = \int_0^1 \int_0^{\sqrt{1-x}} \int_0^{1-x} dy\,dx\,dz$$

Computing this integral is again not too bad:

$$\begin{aligned}
V &= \int_0^1 \int_0^{\sqrt{1-z}} 1 - x\,dx\,dz \\
&= \int_0^1 x - \frac{1}{2}x^2\Big|_{x=0}^{x=\sqrt{1-z}}\,dz \\
&= \int_0^1 \sqrt{1-z} - \frac{1}{2} + \frac{1}{2}z \\
&= -\frac{2}{3}(1-z)^{3/2} - \frac{1}{2}z + \frac{1}{4}z^2\Big|_{z=0}^{z=1} \\
&= \left(0 - \frac{1}{2} + \frac{1}{4}\right) - \left(-\frac{2}{3}\right) = \boxed{\frac{5}{12}}
\end{aligned}$$

again.

- Now let's do $dxdydz$. As before, $0 \le z \le 1$. Now for a fixed $z$, we must find appropriate bounds on $x$ and $y$, where the $x$ bound can involve $y$. This is a bit more difficult than the previous bounds. The base region is a triangle with vertices at $x = 1$, $y = 1$ and the origin; but as $z$ increases, we'll cut off the tip of the triangle (in the $x$ direction) and integrate over a smaller region. In the extreme, when $z = 1$, we're only integrating over a line segment between the origin and the point $(0, 1)$. At height $z$, we'll have a trapezoid with two sides on the axes: The other two sides are described by $x = \sqrt{1-z}$ and $y = 1 - x$. As we're integrating in $x$ first, we have to separate this into two regions: A rectangular region described by

  $$0 \le x \le \sqrt{1-z}, \qquad 0 \le y \le 1 - \sqrt{1-z}$$

  (draw this! The $y$-bound is found by finding $y$ at the extreme right bound) together with a triangle containing the rest of it:

  $$0 \le x \le 1 - y, \qquad 1 - \sqrt{1-z} \le y \le 1$$

  Hence,

  $$V = \int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} dx\,dy\,dz + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} dx\,dy\,dz$$

  The first integral is $1/6$, and the second is $1/4$: adding these gives the correct result of $\boxed{5/12}$.

- For the final two orders, we integrate in $y$ last: The $y$ bounds are $0 \le y \le 1$. Now imagine a fixed $y$; this corresponds to taking a slice of our object along the $xz$-plane (at some displacement $y$). If we integrate in $z$ first, then the bound $0 \le z \le 1 - x^2$ still works; to integrate in $x$, we just rearrange our bound to find $x \le 1 - y$. So we can write the integral as

  $$\begin{aligned}
  V &= \int_0^1 \int_0^{1-y} \int_0^{1-x^2} dz\,dx\,dy \\
  &= \int_0^1 \int_0^{1-y} 1 - x^2\,dx\,dy \\
  &= \int_0^1 x - \frac{x^3}{3}\Big|_{x=0}^{x=1-y}\,dy \\
  &= \int_0^1 (1-y) - \frac{(1-y)^3}{3}dy \\
  &= y - \frac{1}{2}y^2 - \frac{(1-y)^4}{12}\Big|_{y=0}^{y=1} \\
  &= 1 - \frac{1}{2} + 0 - \left(0 - 0 - \frac{1}{12}\right) = \boxed{\frac{5}{12}}
  \end{aligned}$$

- The final ordering, $dxdzdy$ is pretty similar to the ordering $dxdydz$ above. If we fix $y$, then $x$ ranges from 0 to $1-y$, while $z$ ranges from the bottom at $z=0$, to the top curve $1-x^2$. To integrate this *dx first*, we must split this into two regions: A rectangle, and a curved region. The region has vertices at the origin and $x=0, z=1$, as well as $x=1-y, z=0$ and $x=1-y, z=1-x^2=2y-y^2$. Hence, we should have
$$V=\int_0^1\int_0^{2y-y^2}\int_0^{1-y} dx\,dz\,dy+\int_0^1\int_{2y-y^2}^1\int_0^{\sqrt{1-z}} dx\,dz\,dy$$
These integrals are 1/4 and 1/6, respectively - so we again got $\boxed{\frac{5}{12}}$.